

Dynamic Programming (linear)

Step 1: Identify the sub-problem in words.

Step 2: Write out the sub-problem as a recurring mathematical decision.

Step 3: Solve the original problem using Steps 1 and 2.

Step 4: Determine the dimensions of the memoization array and the direction in which it should be filled.

Step 5: Code it! In recursive or iterative way.

E-OLYMP 1560. Decreasing number There are three types of operations you can perform on an integer:

1. If it's divisible by 3, divide it by 3;
2. If it's divisible by 2, divide it by 2;
3. Subtract 1.

Given a positive integer n , find the minimal number of operations needed to produce the number 1.

► Let $f(n)$ contains the minimum number of operations to convert the number n to 1. For example,

- $f(1) = 0$, since we already have number 1;
- $f(2) = 1$, perform operations $2 \rightarrow 1$;
- $f(5) = 3$, perform operations $5 \rightarrow 4 \rightarrow 2 \rightarrow 1$;
- $f(10) = 3$, perform operations $10 \rightarrow 9 \rightarrow 3 \rightarrow 1$;

In the case of $n = 10$ it is better to subtract 1 first than to use the greedy idea and divide by 2.

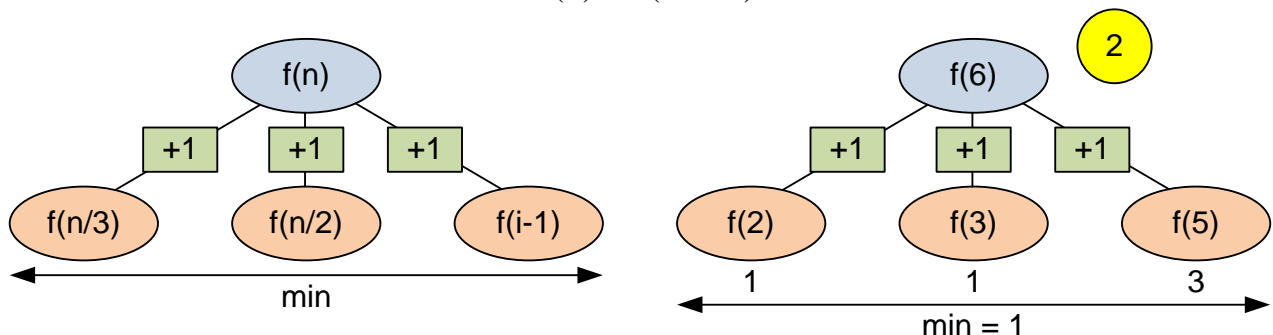
Consider the process of calculating the function $f(n)$.

- If we divide number n by 3 (if n is divisible by 3), then

$$f(n) = f(n / 3) + 1$$
- If we divide number n by 2 (if n is divisible by 2), then

$$f(n) = f(n / 2) + 1$$
- If we subtract 1 from n , then

$$f(n) = f(n - 1) + 1$$



From the number n we can get one of three numbers: $n / 3$, $n / 2$ or $n - 1$. The number of operations for which each of these numbers we can be reduced to 1, equals to $f(n / 3)$, $f(n / 2)$ and $f(i - 1)$ respectively. Since we are interested in the smallest number of operations, we have the relation:

$$f(n) = \min(f(n - 1), f(n / 2), f(n / 3)) + 1, \\ f(1) = 0$$

Moreover, if n is not divisible by 2 (or by 3), then the corresponding element ($f(n / 2)$ or $f(n / 3)$) is absent in the function *min*. For example, for $n = 8$ we have:

$$f(8) = \min(f(7), f(4)) + 1$$

For $n = 7$ we get:

$$f(7) = \min(f(6)) + 1 = f(6) + 1$$

The values of the function $f(n)$ will be stored in the cells of array $d[\text{MAX}]$, where $\text{MAX} = 10^6 + 1$. Fill the cells of array d from 1 to 10^6 according to the given recurrence relation. For example, the following table shows the values of $d[i]$ for $1 \leq i \leq 11$:

i	1	2	3	4	5	6	7	8	9	10	11
$d[i]$	0	1	1	2	3	2	3	3	2	3	4

For example, $d[10] = \min(d[9], d[5]) + 1 = \min(2, 3) + 1 = 3$. It means that it is more efficient to subtract 1 from 10, rather than divide it by 2.

Exercise. Find the values of $d[i]$ for the next i :

i	12	13	14	15	16	17	18	19	20
$d[i]$									

E-OLYMP 1619. House robber You are a professional robber planning to rob houses along a street. Each house has a certain amount of money stashed, the only constraint stopping you from robbing each of them is that adjacent houses have security system connected and it will automatically contact the police if two adjacent houses are broken into on the same night.

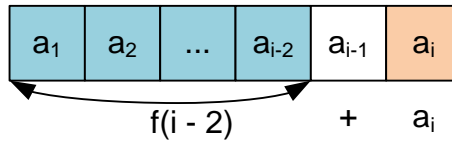
Given a list of non-negative integers representing the amount of money of each house, determine the maximum amount of money you can rob tonight without alerting the police.

► Let's number the houses starting from index one (i -th house contains a_i money). Let $f(i)$ be the maximum amount of money that can be robbed from houses with numbers from 1 till i -th.

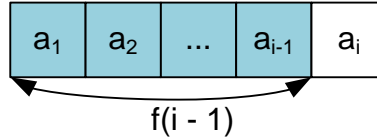
Then $f(1) = a_1$, $f(2) = \max(a_1, a_2)$.

To calculate $f(i)$ we consider two cases:

- If the i -th house is robbed, then one can't rob the $(i - 1)$ -th house. In this case profit will be $f(i - 2) + a_i$.



- if the i -th house is not robbed, the profit will be $f(i - 1)$.



So we have

$$f(i) = \max(f(i - 2) + a_i, f(i - 1))$$

Store the values of $f(i)$ in array `res`. The answer to the problem is the value of $f(n) = \text{res}[n]$.

i	1	2	3	4	5
a_i	6	1	2	10	4
$f(i)$	6	6	8	16	16

6
6

$$f(1) = 6$$

6	1
6	6

$$f(2) = 6$$

6	1	2
6	6	8

$$f(3) = \max(f(2), f(1) + a_3) = \max(6, 8) = 8$$

6	1	2	10
6	6	8	16

$$f(4) = \max(f(3), f(2) + a_4) = \max(8, 16) = 16$$

Let's find $f(3)$. If house 3 is not robbed, the income is $f(2) = 6$. If house 3 is robbed, we can rob first house with income $f(1) = 6$ plus income for the third house that equals to 2 (the total profit is $6 + 2 = 8$).

Let's find $f(4)$. If fourth house is not robbed, the income is $f(3) = 8$. If fourth house is robbed, we can rob the first two houses with an income of $f(2) = 6$ plus income for the fourth house which equals to 10. Equating the total profit to 16 ($6 + 10 = 16$).

Exercise. Find the values of $f(i)$ for the next input data:

i	1	2	3	4	5	6	7
a_i	6	3	2	8	4	1	7
$f(i)$							

E-OLYMP 9036. Dice combinations Your task is to count the number of ways to construct sum n by throwing a dice one or more times. Each throw produces an outcome between 1 and 6.

For example, if $n = 3$, there are 4 ways:

- $1 + 1 + 1$

- $1 + 2$
- $2 + 1$
- 3

► Let $f(n)$ be the number of ways one can get the sum n . Let the number k ($1 \leq k \leq 6$) fell on the last throw. Then all throws except the last one should get the number $n - k$, which can be done in $f(n - k)$ ways. Thus, we have:

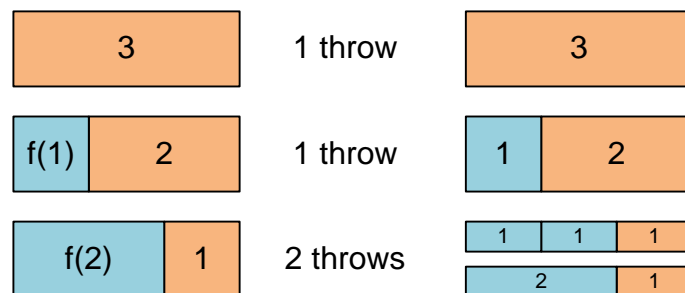
$$f(n) = f(n - 1) + f(n - 2) + f(n - 3) + f(n - 4) + f(n - 5) + f(n - 6)$$

The sum $n = 1$ can only be obtained in one way, by rolling the dice once and getting 1 on it, so $f(1) = 1$.

The sum $n = 2$ can be obtained in two ways: $1 + 1$ and 2 , so $f(2) = 2$.

Let $n = 3$. We can:

- get 3 with one throw;
- get 2 with the last throw and get 1 with the rest throws, which can be done in $f(1) = 1$ way;
- get 1 with the last throw and get 2 with the remaining throws, which can be done in $f(2) = 2$ ways;



Thus $f(3) = 1 + f(1) + f(2) = 1 + 1 + 2 = 4$.

Similarly, for $n \leq 6$ we have: $f(n) = 1 + f(1) + f(2) + \dots + f(n - 1)$.

Consider our recurrence again:

$$f(n) = f(n - 1) + f(n - 2) + f(n - 3) + f(n - 4) + f(n - 5) + f(n - 6)$$

Write the equation for $f(n - 1)$:

$$f(n - 1) = f(n - 2) + f(n - 3) + f(n - 4) + f(n - 5) + f(n - 6) + f(n - 7)$$

From the last equation find the sum

$$f(n - 2) + f(n - 3) + f(n - 4) + f(n - 5) + f(n - 6) = f(n - 1) - f(n - 7)$$

and substitute it into the original recurrence:

$$f(n) = f(n - 1) + f(n - 1) - f(n - 7) = 2 * f(n - 1) - f(n - 7)$$

Compute the values of the function for $n \leq 6$:

$$f(1) = 1, f(2) = 2, f(3) = 4,$$

$$f(4) = 1 + f(1) + f(2) + f(3) = 1 + 1 + 2 + 4 = 8,$$

$$f(5) = 1 + f(1) + f(2) + f(3) + f(4) = 1 + 1 + 2 + 4 + 8 = 16,$$

$$f(6) = 1 + f(1) + f(2) + f(3) + f(4) + f(5) = 1 + 1 + 2 + 4 + 8 + 16 = 32$$

We got the equivalent recurrence:

$$\begin{cases} f(n) = 2f(n-1) - f(n-7), n > 6 \\ f(n) = 2^{n-1}, 1 \leq n \leq 6 \\ f(0) = 1 \end{cases}$$

Compute the values of the function $f(n)$ for $n \leq 9$.

n	0	1	2	3	4	5	6	7	8	9
f(n)	1	1	2	4	8	16	32	63	125	248

For example

$$\begin{aligned} f(8) &= f(7) + f(6) + f(5) + f(4) + f(3) + f(2) = 125, \\ f(9) &= 2 * f(8) - f(2) = 2 * 125 - 2 = 248 \end{aligned}$$

E-OLYMP 987. Nails Some nails are hammered on a straight plank. Any two nails can be joined by a thread. Connect some pairs of nails with a thread, so that to each nail will be tied with at least one thread, and the total length of all threads will be minimal.

► Sort the nail's coordinates in array a . Let $dp[i]$ equals to the minimal total length of all thread, when any two nails starting from the first one (the nails are numbered starting from 1) till i -th are connected with the thread.

If $n = 2$, both nails must be joined with the thread, so

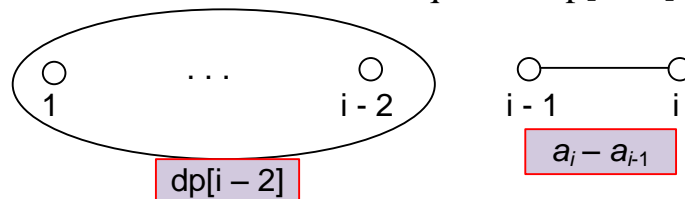
$$dp[2] = a_2 - a_1$$

If $n = 3$, we must connect first nail with the second, and second with the third. So

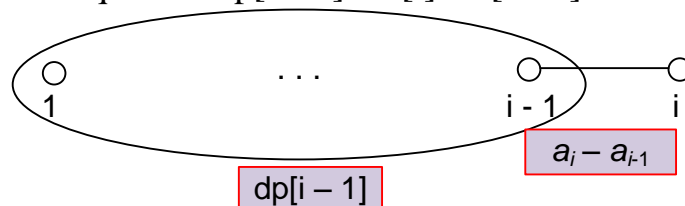
$$dp[3] = a_3 - a_1$$

To add i -th nail one has two possibilities to join it with the thread:

1) connect first $i - 2$ nails among themselves, the $(i - 1)$ -th nail connect to the i -th. The total length of the thread for such connection equals to $dp[i - 2] + a[i] - a[i - 1]$.



2) connect first $i - 1$ nails among themselves, the i -th nail we connect to the $(i - 1)$ -th. The length of the thread equals to $dp[i - 1] + a[i] - a[i - 1]$.

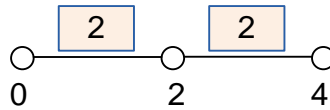


Select the connection method where the total length of the thread is smallest. So

$$dp[i] = \min(dp[i - 2], dp[i - 1]) + a[i] - a[i - 1]$$

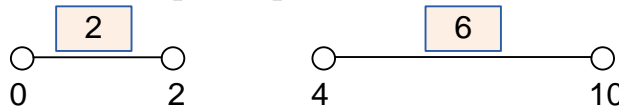
Sort the nail's coordinates for the sample input: 0, 2, 4, 10, 12. For two nails $dp[2] = 2 - 0 = 2$. For three nails we must connect them all with the thread (each nail must be tied with at least one thread), so

$$dp[3] = (4 - 2) + (2 - 0) = 4$$



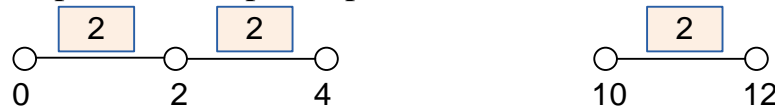
Let's calculate the optimal length of the thread for 4 nails:

$$dp[4] = \min(dp[2], dp[3]) + 10 - 4 = 2 + 6 = 8$$



For 5 nails the minimum possible length of the thread equals to

$$dp[5] = \min(dp[3], dp[4]) + 12 - 10 = 4 + 2 = 6$$



i	1	2	3	4	5
a_i	0	2	4	10	12
$dp[i]$	0	2	4	8	6

Exercise. Find the values of $dp[i]$ for the next input data:

i	1	2	3	4	5	6
a_i	2	4	5	8	10	13
$dp[i]$						

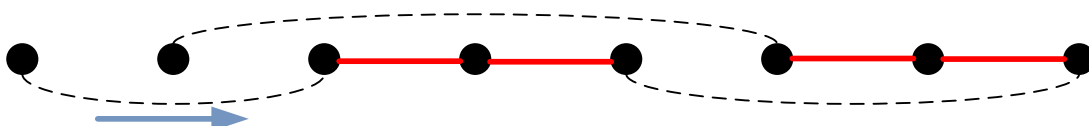
E-OLYMP 8596. Journey from west to east There are n cities standing on a straight line from west to east. The cities are numbered from 1 to n , in order from west to east. Each point on the line has its own one-dimensional coordinate, and the point closer to the east has a large coordinate. The coordinate of the i -th city is x_i .

You are now in city 1, and want to visit all cities. You have two ways to travel:

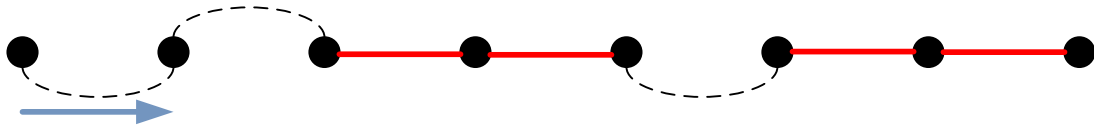
- Walk in a straight line. At the same time, your level of fatigue will increase by a units each time you move a distance of 1, regardless of the direction.
- Teleport to any point you want. Your fatigue level will increase by b units, regardless of teleported distance.

Find the lowest possible level of fatigue, at which you will visit all the cities.

► Consider some optimal (with a minimum level of fatigue) route to visit all cities. It can always be rebuilt so that the movement is carried out from left to right with visits to consecutive cities. For example the following route



can be converted to



with the same level of fatigue.

Let $dp[i]$ be the the minimum level of fatigue with which you can reach city i from city 1 moving sequentially through cities from left to right. It is obvious that $dp[0] = 0$.

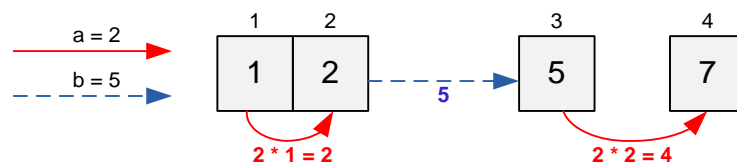
You can get to the i -th city from the $(i - 1)$ -th in two ways:

- walk in a straight line. Then fatigue level will increase by $a * (x[i] - x[i - 1])$;
- teleport. Then fatigue level will increase by b ;

Since overall fatigue should be minimized, it is necessary to choose the path for which the fatigue is minimum. In this way

$$dp[i] = dp[i - 1] + \min(a * (x[i] - x[i - 1]), b)$$

Test 1. From the 1st city we go to the 2nd, after we teleport to the 3rd. At the end we go to the 4th. The fatigue level at the end will be $2 * 1 + 5 + 2 * 2 = 11$, which is the lowest possible.



$$dp[1] = 0;$$

$$dp[2] = dp[1] + \min(a * (2 - 1), b) = 0 + \min(2 * (2 - 1), 5) = 0 + 2 = 2;$$

$$dp[3] = dp[2] + \min(a * (5 - 2), b) = 2 + \min(2 * (5 - 2), 5) = 2 + 5 = 7;$$

$$dp[4] = dp[3] + \min(a * (7 - 5), b) = 7 + \min(2 * (7 - 5), 5) = 7 + 4 = 11;$$

i	1	2	3	4
a_i	1	2	5	7
$dp[i]$	0	2	7	11

Test 2. From city 1 we just go to all cities up to 7th. As a result, the fatigue level will be 84, which is the lowest possible.

Test 3. Visit all cities, in any order, teleporting six times. The fatigue level will be 12, which is the lowest possible.

Exercise. Find the values of $dp[i]$ for the next input data:

i	1	2	3	4	5	6
a_i	1	3	6	10	11	13
$dp[i]$						

E-OLYMP 4051. Grasshopper Grasshopper lives in the teacher's room. It likes to jump on one dimensional checkerboard. The length of the board is n cells. To its regret, it can jump only on $1, 2, \dots, k$ cells forward.

Once teachers wondered in how many ways a grasshopper can reach the last cell from the first one. Help them to answer this question.

► Let $dp[i]$ equals to the number of ways for grasshopper to leap from the first cell to the i -th one. Set $dp[1] = 1, dp[2] = 1$.

If $2 < i \leq k$, then its possible to get into the i -th cell from any previous one, so

$$dp[i] = dp[1] + dp[2] + \dots + dp[i-1] = \sum_{j=1}^{i-1} dp[j]$$

Let k be large, calculate $dp[i]$ using this formula:

$$\begin{aligned} dp[3] &= dp[1] + dp[2] = 1 + 1 = 2, \\ dp[4] &= dp[1] + dp[2] + dp[3] = 1 + 1 + 2 = 4, \\ dp[5] &= dp[1] + dp[2] + dp[3] + dp[4] = 1 + 1 + 2 + 4 = 8 \end{aligned}$$

You can notice that $dp[i] = 2 * dp[i-1]$. However, this formula can be obtained from the following considerations. From

$$dp[i-1] = dp[1] + dp[2] + \dots + dp[i-2]$$

follows that

$$\begin{aligned} dp[i] &= (dp[1] + dp[2] + \dots + dp[i-2]) + dp[i-1] = \\ &= dp[i-1] + dp[i-1] = 2 * dp[i-1] \end{aligned}$$

If $i > k$, then its possible to get into the i -th cell from any out of k previous, so

$$dp[i] = \sum_{j=i-k}^{i-1} dp[j] = dp[i-k] + \dots + dp[i-1]$$

Similarly, you can see that from the fact that

$$dp[i-1] = dp[i-k-1] + \dots + dp[i-2]$$

follows that

$$\begin{aligned} dp[i] &= dp[i-k] + \dots + dp[i-1] = \\ &= (dp[i-k-1] + dp[i-k] + \dots + dp[i-2]) + dp[i-1] = \\ &= 2 * dp[i-1] - dp[i-k-1] \end{aligned}$$

For the given sample test $n = 8$ and $k = 2$ the state of dp array has the from:

i	1	2	3	4	5	6	7	8
dp[i]	1	1	2	3	5	8	13	21

For $n = 8$ and $k = 4$ the array dp has the form:

i	1	2	3	4	5	6	7	8
dp[i]	1	1	2	4	8	15	29	56

$$\begin{aligned} \leftarrow k = 4, s = 8 \rightarrow \\ \leftarrow s = s - 1 + 8 = 15 \rightarrow \\ \leftarrow s = s - 1 + 15 = 29 \rightarrow \\ \leftarrow s = s - 2 + 29 = 56 \rightarrow \end{aligned}$$

Exercise. Fill dp array for $n = 8$ and $k = 3$.

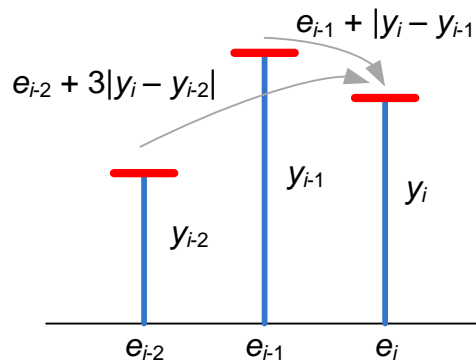
i	1	2	3	4	5	6	7	8
dp[i]								

E-OLYMP 798. Platforms In older games one can run into the next situation. The hero jumps along the platforms that hang in the air. He must move himself from one side of the screen to the other. When the hero jumps from one platform to the neighboring, he spends $|y_2 - y_1|$ energy, where y_1 and y_2 are the heights where these platforms hang. The hero can make a super jump that allows him to skip one platform, but it takes him $3 \cdot |y_3 - y_1|$ energy.

You are given the heights of the platforms in order from the left side to the right. Find the minimum amount of energy to get from the 1-st (start) platform to the n -th (last). Print the list (sequence) of the platforms that the hero must pass.

► Let $e[i]$ contains the minimum amount of energy sufficient to get from platform 1 to platform i . Obviously, $e[1] = 0$ (to get from the first platform to the first is zero energy), and $e[2] = |y_2 - y_1|$, since the second platform can only be reached from the first one.

In cell $p[i]$, we'll store the number of the platform from which we jumped to the i -th. Initially, we set $p[1] = -1$ (initially we are on the first platform), and also $p[2] = 1$.

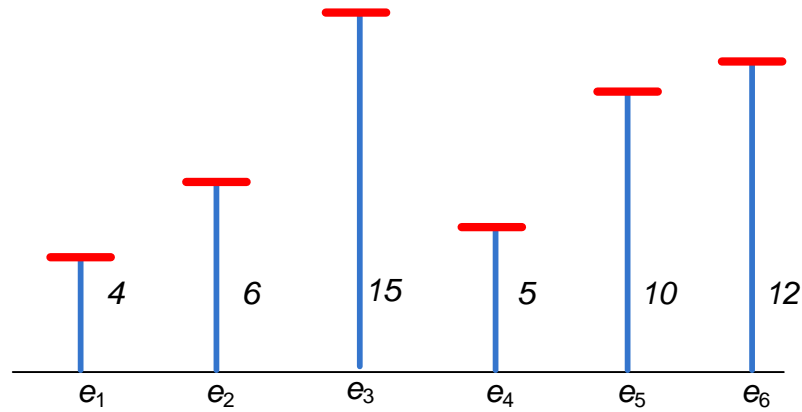


To the i -th platform ($i \geq 3$) you can jump either from $(i - 1)$ -th, spending $e[i - 1] + |y_i - y_{i-1}|$ energy, or from $(i - 2)$ -th, having made a super jump and spending $e[i - 2] + 3 \cdot |y_i - y_{i-2}|$ energy. So

$$e[i] = \min(e[i - 1] + |y_i - y_{i-1}| , e[i - 2] + 3 \cdot |y_i - y_{i-2}|)$$

If to the i -th platform the jump is performed from $(i - 1)$ -th, then set $p[i] = i - 1$. If from $(i - 2)$ -th, then we set $p[i] = i - 2$. To find the number of platforms on which jumps from the first to the n -th were performed, one should walk from the n -th platform to the first one each time moving from the i -th platform to $p[i]$ -th.

Let we have 6 platforms with heights 4, 6, 15, 5, 10, 12.

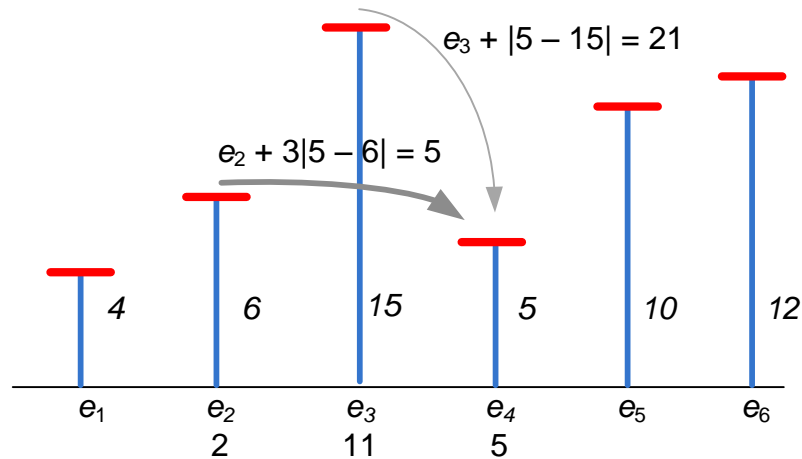


Calculate the spent energy when jumping:

$$e[1] = 0, p[1] = -1;$$

$$e[2] = |6 - 4| = 2, p[2] = 1;$$

$$e[3] = \min(e[2] + |15 - 6|, e[1] + 3 * |15 - 4|) = \min(11, 33) = 11, p[3] = 2;$$



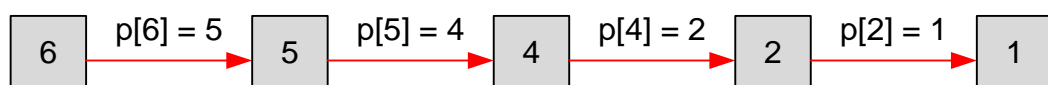
$$e[4] = \min(e[3] + |5 - 15|, e[2] + 3 * |5 - 6|) = \min(21, 5) = 5, p[4] = 2;$$

$$e[5] = \min(e[4] + |10 - 5|, e[3] + 3 * |10 - 15|) = \min(10, 26) = 10, p[5] = 4;$$

$$e[6] = \min(e[5] + |12 - 10|, e[4] + 3 * |12 - 5|) = \min(12, 26) = 12, p[6] = 5;$$

i	1	2	3	4	5	6
y_i	4	6	15	5	10	12
e_i	0	2	11	5	10	12
p_i	-1	1	2	2	4	5

To restore the path, move from the final (6-th) platform back along the indexes of $p[i]$:



The list of platforms to go through is as follows:

1, 2, 4, 5, 6

Exercise. Fill the arrays with the next input data.

i	1	2	3	4	5	6
y_i	8	4	1	5	12	3
e_i						
p_i						